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MODIFIED NEUBER–NOVOZHILOV CRITERION OF RUPTURE FOR V-SHAPED CUTS (ANTIPLANE PROBLEM)

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A modified Neuber–Novozhilov criterion of rupture is proposed to study the fracture in the vicinity of the tip of a V-shaped cut. In this criterion, the limits of averaging stresses at the cut axis depend on the presence, size, and location of defects in the initial material. The lattice parameter of the initial material is taken to be the characteristic linear size. For V-shaped cuts, simple relations between the stress-intensity factor of the modified singularity factor, the singularity factor itself, and the theoretical shear strength of a single crystal of the material with allowance for damage in the vicinity of the tip are proposed. These relations admit passage to the limit with respect to the angle from a V-shaped cut to a crack. It is shown that the classical critical stress-intensity factor used to estimate the strength of cracked bodies is not a material constant.

Introduction. The Neuber–Novozhilov approach [1, 2] allows one to describe the fracture of cracked media with hierarchic structures [3–5] under loads corresponding to three classical types of cracks. The author [3–5] considered sharp cracks modeled by bilateral cuts and blunt cracks shaped like narrow cuts with rounded tips and parallel flanks. The discrete–integral criteria were constructed with the use of the concepts of classical fracture mechanics (solid mechanics) and solid state physics [6, 7] related to the crystal structure of single crystals. If actual spatial arrangement of atoms in a single crystal is taken into account, the cracks in the crystal cannot be modeled by bilateral cuts. Even in the two-dimensional case, it makes sense to consider V-shaped cuts with the opening angle of the cut determined by characteristics of the crystal lattice. Specific features of the fracture problems of solids with sharp V-shaped cuts were considered in [8, 9]. The stress fields in the vicinity of a V-shaped cut consist of regular and singular components with the singularity factor dependent on the opening angle of the cut [10]. The singularity factor coincides with the singularity factor of the stress field in the vicinity of the crack tip only in the limiting case, where the V-shaped cut in the vicinity of the tip becomes a bilateral cut (crack). The stress field in the vicinity of the V-shaped cut has the simplest form for antiplane strain. The criterion of brittle fracture proposed below can be used to estimate the strength of twisted shafts with cracks (necks) if the conditional regular granularity of the shaft material is known.

1. Stresses in the Vicinity of the Tip of a V-Shaped Cut. We consider the stress field in the vicinity of the tip of a V-shaped cut in the antiplane problem where the stress–strain state does not depend on the third coordinate z (Fig. 1). In Fig. 1, the following notation is used: Oxy and $Or\theta$ are the Cartesian and polar coordinate systems, respectively, β is the half the opening angle of the cut (the axis of the cut coincides with the Ox axis); moreover, $\alpha + \beta = \pi$, where $\beta > 0$ for $\alpha < \pi$ and $\beta < 0$ for $\alpha > \pi$. We assume that a solid symmetric about the cut axis is loaded symmetrically about the cut. Hence, owing to the symmetry of the problem, the maximum stresses occur at the cut axis. In the vicinity of the V-shaped-cut tip, the shear stresses $\tau_{\theta z}(r, \theta)$ at the cut axis $\theta = 0$ can be written with accuracy to higher-order terms of the linear problem in the form

$$\tau_{\theta z}(r, 0) \simeq \tau_\infty + K_{\rm III} r^{\omega - 1}/(2\pi)^{1/2}. \qquad (1.1)$$

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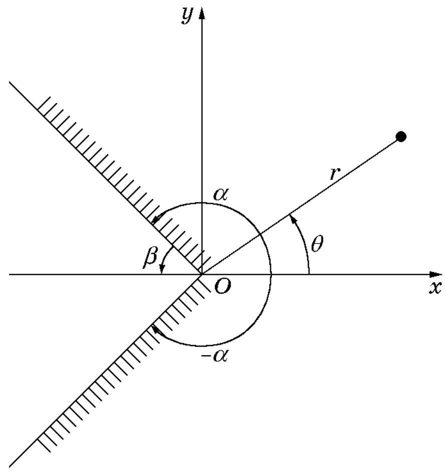

Fig. 1.

Here $\tau_\infty = \text{const}$ are the characteristic stresses, $K_{\rm III}$ is the generalized stress-intensity factor (SIF) for the antiplane problem for the singular component $r^{\omega-1}$, and $\omega = \omega(\alpha)$ are the roots of the characteristic equation [10]

$$\cos(\alpha\omega) = 0. \tag{1.2}$$

The characteristic stresses τ_∞ are determined for the constructed stress field $\tau_{\theta z}(r,\theta)$. It is noteworthy that, in the general case, construction of the stress field $\tau_{\theta z}(r,\theta)$ is a rather complex separate problem. In the limit as $\beta \to 0$, the V-shaped cut becomes a bilateral cut and $\omega = 1/2$.

Of the set of the solutions of Eq. (1.2), only the solution corresponding to the first positive root has a mechanical meaning: $\omega = \pi/(2\alpha)$. We consider three cases: $\omega > 1/2$ for $\alpha < \pi$, $\omega < 1/2$ for $\alpha > \pi$, and $\omega = 1/2$ for a crack (bilateral cut) for $\alpha = \pi$. For a crack, the generalized SIF ($K_{\rm III}$) becomes the classical SIF ($K^0_{\rm III}$) for a sharp crack. We note that methods of classical fracture mechanics of cracked solids are applicable only in the last case [8, 9]. In the first case, the singularity of the stress field is smaller than the singularity of the stress field at the crack tip; in the second case, it is greater, and, hence, the classical approach is not applicable for the strength analysis of solids with V-shaped cuts [see (1.1)]. Thus, we have $1/3 < \omega < 1$ for $\pi/2 < \alpha < 3\pi/2$. The case where $\alpha = \pi/2$ since $\omega = 1$ corresponds to a half-plane; the singular component vanishes in this case.

2. Criterion of Brittle Fracture of Solids with V-shaped Cuts. We study single crystals with V-shaped cuts whose opening angles are determined by characteristics of the crystal lattice. We confine our attention to the two-dimensional case. It is assumed that there are vacancies ahead of the cut tip. Figure 2 shows a closely packed atomic layer with a vacancy (the atoms are shown by circles, the vacancy is denoted by the cross, r_e is the interatomic distance, and $\beta = \pi/3$).

We propose the following discrete–integral criterion of brittle strength (two-dimensional case) of the weakest monoatomic layer for V-shaped cuts in the antiplane problem:

$$\frac{1}{kr_e}\int\limits_0^{nr_e} \tau_{\theta z}(r,0)\,dr \leqslant \tau_m. \tag{2.1}$$

Here $\tau_{\theta z}(r,0)$ is the shear stress at the cut axis (this stress acts deep within the single crystal), n and k are integers ($n \geqslant k$), where k is the number of interatomic bonds and nr_e is the interval of averaging (in the case shown in Fig. 2, we have $n = 2$ and $k = 1$), and τ_m is the theoretical (ideal) shear strength of the single crystal in the plane $\theta = 0$.

After the corresponding transformations [see (1.1) and (1.2)], for the antiplane problem, we obtain the following estimate of the generalized SIF $K_{\rm III}$ for the sharp V-shaped cut in the presence of vacancies at its axis:

$$\frac{K_{\rm III}}{\omega(2\pi)^{1/2}(nr_e)^{1-\omega}}\,\frac{1}{\tau_\infty} \leqslant \frac{\tau_m}{\tau_\infty}\,\frac{k}{n} - 1. \tag{2.2}$$

As $\beta \to 0$, we have $\omega = 1/2$, and estimate (2.2) becomes the estimate for the classical SIF $K^0_{\rm III}$ of a sharp crack:

$$2K^0_{\rm III}/(\tau_\infty\sqrt{2\pi nr_e}) \leqslant (\tau_m/\tau_\infty)(k/n) - 1. \tag{2.3}$$

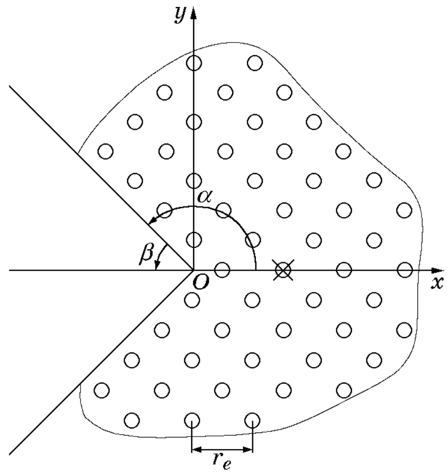

Fig. 2

For $K_{\rm III}^0 = K_{\rm III}^{0*}$, the last inequality becomes an equality ($K_{\rm III}^{0*}$ is the critical SIF for a crack in classical fracture mechanics). Since the SIFs of sharp internal and edge cracks are $K_{\rm III}^0 = \tau_\infty\sqrt{\pi l_{nk}}$ and $K_{\rm III}^0 = 1.1215\tau_\infty\sqrt{\pi l_{nk}}$, respectively ($2l_{nk}$ and l_{nk} are the lengths of the internal and edge cracks, respectively) (see [11]), the critical lengths of these cracks $2l_{nk}^*$ and l_{nk}^* satisfy the equalities

$$\frac{2l_{nk}^*}{r_e} = \Big(\frac{\tau_m}{\tau_\infty} - \frac{n}{k}\Big)^2 \frac{k^2}{n}, \qquad 2.52\,\frac{l_{nk}^*}{r_e} = \Big(\frac{\tau_m}{\tau_\infty} - \frac{n}{k}\Big)^2 \frac{k^2}{n}. \tag{2.4}$$

One can see that relations (2.3) and (2.4) admit the limiting passage as $K_{\rm III}^0 \to 0$ and $l_{nk} \to 0$. In the absence of microdefects (vacancies) and macrodefects (cracks) in a specimen, we have $n = k = 1$ and $l_{nk} = 0$, respectively. In this case, the theoretical strength of an ideal crystalline material τ_m is reached.

It is noteworthy that there exist exact limiting relations [11] for determining the SIFs of sharp cracks in terms of the stress-concentration coefficients at the tip of a narrow cut. Stress-concentration coefficients [1] have always been related to the geometrical parameters of the problem studied, whereas the critical SIF for a crack $K_{\rm III}^{0*}$ in classical fracture mechanics is assumed to be a material constant.

We call attention to strange dimensionality of the generalized SIF $K_{\rm III}$, which depends on the opening angle of the cut [see (2.2)]. According to the concepts of classical fracture mechanics, the critical generalized SIF of a material $K_{\rm III}^*$ depends on the opening angle of the cut. Figure 3 shows the shear stress at the axis of a cut or crack $\tau_{\theta z}$ versus the x coordinate for increasing $K_{\rm III}$: curves 2 and 5 refer to the stress distribution for a crack ($\alpha = \pi$), where $K_{\rm III}^{(2)} > K_{\rm III}^{(5)}$ (the dashes at curve 2 show that the classical SIF reaches the critical value for the material considered, i.e., $K_{\rm III}^{(2)} = K_{\rm III}^{0*}$); curves 1 and 4 refer to the stress distribution for a cut with $\pi/2 < \alpha < \pi$ and $K_{\rm III}^{(1)} > K_{\rm III}^{(4)}$; curves 3 and 6 refer to the stress distribution for a cut with $\alpha > \pi$ and $K_{\rm III}^{(3)} > K_{\rm III}^{(6)}$. According to the discrete–integral criterion (2.1), the critical state of a crystal structure ahead of the tip of a crack or cut occurs when the averaged stresses in the interval $(0, nr_e)$ reach the theoretical strength with allowance for the damages of the material. Criterion (2.1) is the force criterion in the interval $(0, nr_e)$. The minimum length of the averaging interval is r_e. Estimate (2.2), which takes into account the material structure, is a local estimate determined mainly by the singularity factor $1-\omega$ (Fig. 3). It is known in classical linear fracture mechanics that the criterion of crack initiation according to the critical SIF (force criterion) is equivalent to the energy criterion of fracture [12].

In classical fracture mechanics, one needs to determine the critical value of the generalized SIF $K_{\rm III}^* = K_{\rm III}^*(\alpha)$ for each material and each angle α. It makes sense to assume that the critical SIF for a crack is not a material constant. The theoretical strength τ_m of the regular structure considered is a material constant, which is seen from formula (2.2), and the generalized SIF $K_{\rm III}$ constructed according to the given boundary conditions [see (2.2)] is a convenient approximation of solution (1.1).

3. Stress-Concentration Coefficient in a Structured Material for Cracks and Necks. We describe stress concentration of shafts [1] whose material has a granular structure. Let the regular structure of the material be described by one linear parameter denoted by r_1. It is assumed that the shaft has a sharp or blunt crack normal

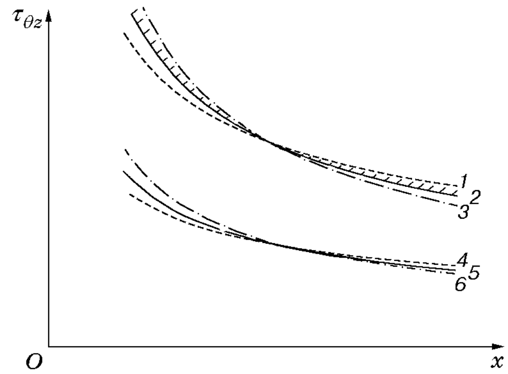

Fig. 3

to its surface. The blunt crack is understood as a narrow cut with a rounded tip of radius ρ. The disruption of the regular structure of the shaft material in the vicinity of the crack tip is ignored.

For out-of-plane shear, the stress distributions relative to the cut axis have the following form (see [11, relation (1.38) and Fig. 1.10]):

$$\tau_{xz} \simeq \tau_\infty + \frac{K^0_{\rm III}}{(2\pi r)^{1/2}} \sin\frac{\theta}{2}, \qquad \tau_{yz} \simeq \tau_\infty + \frac{K^0_{\rm III}}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}. \tag{3.1}$$

Here τ_{xz} and τ_{yz} are the shear stresses, $\tau_\infty = \text{const}$ are the characteristic (nominal) stresses, and $K^0_{\rm III}$ is the classical SIF. In (3.1), the right tip of the sharp crack ($\rho \equiv 0$) or blunt crack ($\rho \neq 0$) coincides with the origin of the polar coordinate system $Or\theta$.

Since the loading is symmetrical, the maximum stress-concentration coefficient occurs at the axis of the cut (crack). Following [1] and taking into account the material structure, we obtain the average value of stresses τ_a in the grain located at the tip of the cut (crack) ($\theta = 0$ and $\rho/2 < r < \rho/2 + r_1$):

$$\tau_a = \frac{1}{r_1} \int\limits_{\rho/2}^{\rho/2+r_1} \tau_{yz}(r, 0)\, dr. \tag{3.2}$$

Remark 1. Relation (3.2) ignores the damage of the material ahead of the crack tip and the structure of grain boundaries at the axis of the cut (crack) (see [1, relation (6) and Fig. 86]).

After some manipulations, we obtain the stress-concentration coefficient $æ = \tau_a/\tau_\infty$ at the tip of the cut (crack), which depends explicitly on the rounding radius of the cut ρ and classical SIF $K^0_{\rm III}$, i.e., $æ = æ(\rho, K^0_{\rm III})$. Expressing the classical SIF in terms of the length l of the sharp edge crack, i.e., $K^0_{\rm III} = 1.1215\tau_\infty\sqrt{\pi l}$, we finally obtain

$$æ = 1 + 1.1215\sqrt{\frac{2l}{r_1}}\Big(\sqrt{\frac{\rho}{2r_1} + 1} - \sqrt{\frac{\rho}{2r_1}}\Big), \quad \rho \geqslant 0. \tag{3.3}$$

In contrast to Neuber's relation [1], formula (3.3) relates explicitly the stress-concentration coefficient to the rounding radius ρ of the cut (crack).

A shaft with a cut (crack) does not fail if the stresses τ_a acting on a finite-size element (grain) r_1 do not exceed the theoretical strength of a structured body τ^*, i.e., $\tau_a = æ\tau_\infty \leqslant \tau^*$. With accuracy to notation, the last inequality coincides with relation (2.1). It makes sense to consider the fracture of a solid with structural hierarchy where $r_e \ll r_1$ [4, 5]. For an ideal crystalline solid, the theoretical (ideal) strength of a single crystal τ_m is well understood [7], whereas the theoretical strength τ^* of a solid of an ideal structure with the characteristic linear grain size r_1 has yet to be studied.

Within the approach proposed, it is possible to demonstrate a unified approach to estimation of the strength of solids with hierarchic structures both in the presence of sharp cracks or cuts (blunt cracks) and in the absence of macrodefects.

TABLE 1

l	ρ	æ	1/æ
$r_e/2$	$r_e/2$	1.69	0.590
r_e	r_e	1.82	0.550
$3r_e/2$	$r_e/2$	2.19	0.457
$2r_e$	r_e	2.16	0.463
$5r_e/2$	$r_e/2$	2.55	0.390
$10r_e$	r_e	5.59	0.278

4. The Effect of the Single-Crystal Surface Layer on the Estimate of Theoretical Strength. The calculated and experimental values of the theoretical (ideal) strength of solids were compared in [7], where the calculations were performed for the internal volumes of ideal single crystals. Table 1 lists the stress-concentration coefficients æ for a single crystal with all imperfections concentrated on its surface. The imperfections are due to the absence of some atoms in the first, second, and third near-surface layers of closely-packed atoms (two-dimensional case). If the imperfections are modeled by a cut (crack), one can calculate the stress-concentration coefficient æ by formula (3.3) for given interatomic distance r_e, length of the cut (crack) l, and rounding radius ρ. The values of the parameter 1/æ, which characterizes the reachability of the theoretical strength of a single crystal τ_m [7], are also listed in Table 1. An analysis of the estimates obtained shows that $(0.5–0.6)\tau_m$ can be obtained only for a single crystal with a nearly ideal surface layer. In the presence of crack-like surface defects, the strength of the single crystal decreases abruptly.

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